AIR FORCE REPORT NO.
SAMSO-TR-68-251

AEROSPACE REPORT NO.
TR-0158(S3960-30)-2

AD 672760

A Method for Determining Probability Distributions for Mass Properties of Systems

by

Frank C. Bond

JUNE 1968

Prepared for
SPACE AND MISSILE SYSTEMS ORGANIZATION
AIR FORCE SYSTEMS COMMAND
Air Force Unit Post Office
Los Angeles, California 90045

AEROSPACE CORPORATION

San Bernardino Operations

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FOREWORD

This report by The Aerospace Corporation, San Bernardino Operations has been done under Contract No. F04695-67-C-0155 as TR-0158(S3960-30)-2. The Air Force program monitor is Major R. Smith, USAF (SMQVB). The dates of research for this report include the period October 1967 through April 1968. This report was submitted by the author in May 1968.

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This technical report has been reviewed and is approved.

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UNCLASSIFIED ABSTRACT

A METHOD FOR DETERMINING PROBABILITY DISTRIBUTIONS FOR MASS PROPERTIES OF SYSTEMS, by Frank C. Bond

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An analysis is performed to develop expressions for the statistical parameters of system mass properties. It is assumed that the component weights, centers of gravity, and inertias have known probability distributions. The general forms of the density functions generated are deduced from the calculus of probability whenever applicable. Means and variances of the system mass properties are computed from linear functions of the statistical parameters of the components. The input random variables may be distributed according to any probability law or combination of probability laws for which the moments are known. From conceptual design to detailed specification drawings, the probability laws governing mass properties estimation change. This method may be utilized at any point in the life cycle of system design as it is independent of the particular forms of the distributions used as input. (Unclassified Report)

CONTENTS

FIGURES

NOMENCLATURE

$Cov[\cdot,\cdot]$	signifies covariance of the random variables $(\cdot,\cdot)$
D_{xy}	$I_{yy} - I_{xx}$
D_{xz}	$I_{zz} - I_{xx}$
$E[\cdot], \mu_{(\cdot)}$	signifies mean of the random variable $(\cdot)$
F_{yz}	$D_{xy} D_{xz}$
H_{yz}	$I_{xy} I_{xz}$
$I_{ox_i}, I_{oy_i}, I_{oz_i}$	moments of inertia of i^{th} component about $X_i\ Y_i\ Z_i$ (see Figure 1)
$I_{oxy_i}, I_{oxz_i}, I_{oyz_i}$	products of inertia of i^{th} component in reference frame $x_i\ y_i\ z_i$
I_{xx}, I_{yy}, I_{zz}	system moments of inertia about axes X, Y, and Z
I_{xy}, I_{xz}, I_{yz}	system products of inertia in reference frame XYZ
m	number of random samples in simulation
n	number of components in system
$SD[\cdot]$	$\sqrt{V[\cdot]}$
t	$\sum_{i=1}^{n} w_i x_i$
t_i	$w_i x_i$
$V[\cdot], \sigma_{(\cdot)}^2$	signifies variance of the random variable $(\cdot)$
w_i	weight of the i^{th} component
W	total system weight $\left(\sum_{i=1}^{n} w_i\right)$

NOMENCLATURE (Continued)

x_i, y_i, z_i	position of center of gravity (cg) of i^{th} component in reference frame XYZ (see Figure 1)
$\bar{x}$, $\bar{y}$, $\bar{z}$	position vectors of system cg in reference frame XYZ
θ	angle between the X axis and that principal axis having the least moment of inertia
θ_{xy}, θ_{xz}	projected angles of principal axis offset angle from the reference axis, X, in the XY and XZ planes respectively
$\hat{\mu}$	estimate of μ from simulation
$\hat{\sigma}^2$	estimate of variance from simulation

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SECTION I

INTRODUCTION

The mass properties of a system are functions of the mass properties of the constituents comprising that system. If the component properties are known, then the system properties are uniquely determined. Consider a system having components whose mass properties are random variables. If these random variables are completely defined, then the system mass properties are random variables having unique probability laws. The problem here is to determine these system mass properties probability laws given the statistical distributions of the components.

In this paper, a model describing the statistical nature of mass properties of systems is developed. Given the probability laws governing the weights, centers of gravity (cg), and inertias of the individual components, the means and variances of the system mass properties may be computed. Complete information about these laws is not required; only the moments of the input random variables are utilized as input. Appendix A lists some of the well-known probability laws with their associated distributions and moments.

A Monte Carlo simulation (Appendix B) was used to verify the theoretically derived expressions for mean and variance for all system mass properties of a typical reentry vehicle. The component mass properties were all input as uniform random variables for convenience. The theoretical results were well within the errors associated with estimating parameters from finite samples (a sample size of 100 was used). Furthermore, all mass properties appeared to be approximately normally distributed regardless of whether the Central Limit Theorem could be applied.

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SECTION II

THE MASS PROPERTIES MODEL

Consider a physical system having n components. The weight of the i^{th} component is w_i. The center of gravity of the i^{th} component is located at (x_i, y_i, z_i) within the system reference frame X Y Z (see Figure 1). Let $X_i\ Y_i\ Z_i$ be a rectangular coordinate system whose axes are parallel to X Y Z and whose origin is at the center of gravity of the i^{th} component. The moments and products of inertia of w_i with respect to $X_i\ Y_i\ Z_i$ are I_{ox_i}, I_{oy_i}, I_{oz_i} and I_{oxy_i}, I_{oxz_i}, I_{oyz_i}, respectively.

Using these component mass properties, the system mass properties with respect to the arbitrary reference frame, XYZ, may be computed as follows:

<u>Weight</u>

$$w = \sum_{i=1}^{n} w_i$$

<u>Center of Gravity, x Coordinate</u>

$$\bar{x} = \frac{\sum_{i=1}^{n} w_i\, x_i}{\sum_{i=1}^{n} w_i}, \quad \text{similarly for } \bar{y} \text{ and } \bar{z}$$

<u>Moment of Inertia about X</u>

$$I_{xx} = \sum_{i=1}^{n} w_i \left(y_i^2 + z_i^2\right) + \sum_{i=1}^{n} I_{ox_i}, \quad \text{similarly for } I_{yy} \text{ and } I_{zz}$$

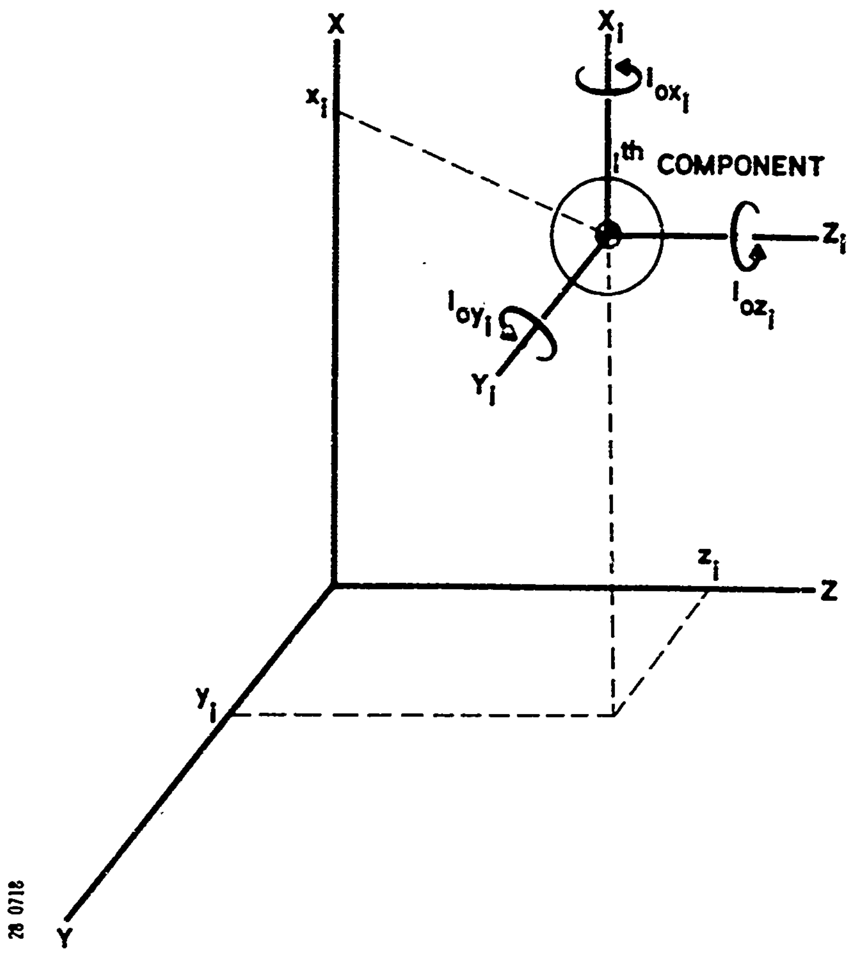

Figure 1. Component Reference Frame

Product of Inertia in XY Plane

$$I_{xy} = \sum_{i=1}^{n} w_i \, x_i \, y_i + \sum_{i=1}^{n} I_{oxy_i} \, , \quad \text{similarly for } I_{xz} \text{ and } I_{yz}$$

Projection of Principal Axis Offset Angle in the XY Plane (See Figure 2)

$$\theta_{xy} = \frac{1}{2} \tan^{-1} \frac{2 \, I_{xy}}{I_{yy} - I_{xx}} \, , \quad \text{similarly for } \theta_{xz}$$

Principal Axis Offset Angle

$$\theta = \tan^{-1}\left(\sqrt{\tan^2 \theta_{xy} + \tan^2 \theta_{xz}}\right)$$

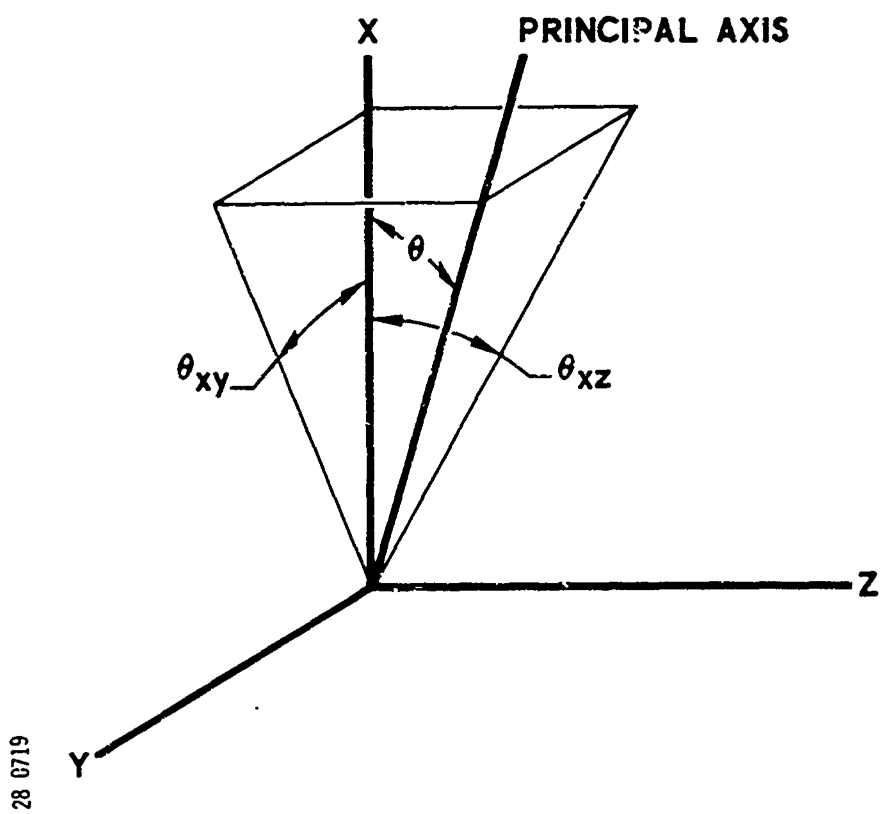

Figure 2. Principal Axis Offset Angle

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SECTION III

PROBLEM STATEMENT

Determine the probability density functions of the following system mass properties:

Weight	w
Components of Center of Gravity	$\bar{x}$, $\bar{y}$, $\bar{z}$
Moments of Inertia	I_{xx}, I_{yy}, I_{zz}
Products of Inertia	I_{xy}, I_{xz}
Components of Principal Axis Offset Angle	θ_{xy}, θ_{xz}
Principal Axis Offset Angle	θ

In particular, for each of the above random variables, determine the mean (μ), variance $\left(\sigma^2\right)$, and the general form of the probability density function (normal, uniform, etc.).

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SECTION IV

RESULTS

A. GENERAL FORMS OF THE SYSTEM MASS PROPERTIES DISTRIBUTION FUNCTIONS

For sufficiently large n, the Central Limit Theorem states that the weight and the moments and products of inertia are approximately normally distributed. Since the equations for the center of gravity and principal axis offset involve dependent terms, the theorem cannot be applied to these latter mass properties. However, the simulation (Appendix B) verifies that the dependency of these terms is weak enough for them to still be normally distributed for most practical purposes. A good rule to follow when no other information is available is to assume that all sums of random variables are normally distributed. If a more precise density function is required in order to make probability statements, simulation may be used.

B. THEORETICALLY DERIVED EXPRESSIONS FOR MEANS (μ) AND VARIANCES (σ^2)

The following results are in terms of the first four moments $\left(E[\cdot],\ E[\cdot^2],\ E[\cdot^3],\ E[\cdot^4]\right)$ of the component mass properties. These means and variances are independent of the forms of the system distribution functions.

Weight, w

$$\mu_w = \sum_{i=1}^{n} E\left[w_i\right]$$

$$\sigma_w^2 = \sum_{i=1}^{n} \left(E\left[w_i^2\right] - E^2\left[w_i\right]\right)$$

X Coordinate of Center of Gravity, $\overline{x}$

$$\mu_{\overline{x}} \cong \frac{\sum_{i=1}^{n} E\left[w_i\right] E\left[x_i\right]}{\sum_{i=1}^{n} E\left[w_i\right]}$$

$$\sigma_{\overline{x}}^{2} \cong \frac{1}{\mu_w^{2}} \left(\sigma_t^{2} + \mu_{\overline{x}}^{2}\, \sigma_w^{2} - 2\mu_{\overline{x}} \operatorname{Cov}\left[t, w\right]\right)$$

where

$$\sigma_t^{2} = \sum_{i=1}^{n} \left(E\left[w_i^{2}\right] E\left[x_i^{2}\right] - E^{2}\left[w_i\right] E^{2}\left[x_i\right]\right)$$

and

$$\operatorname{Cov}\left[t, w\right] = \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} E\left[w_i\right] E\left[x_i\right] E\left[w_j\right]$$

$$+ \sum_{i=1}^{n} E\left[w_i^{2}\right] E\left[x_i\right] - \mu_w \left(\sum_{i=1}^{n} E\left[w_i\right] E\left[x_i\right]\right)$$

Replacing x_i by y_i and z_i, we get the expressions for the means and variances of $\overline{y}$ and $\overline{z}$ respectively.

Moment of Inertia about X, I_{xx}

$$\mu_{I_{xx}} = \sum_{i=1}^{n} E\left[w_i\right]\left(E\left[y_i^2\right] + E\left[z_i^2\right]\right) + \sum_{i=1}^{n} E\left[I_{ox_i}\right]$$

$$\sigma_{I_{xx}}^2 = \sum_{i=1}^{n} E\left[w_i^2\right]\left(E\left[y_i^4\right] + 2\,E\left[y_i^2\right] E\left[z_i^2\right] + E\left[z_i^4\right]\right)$$

$$- \sum_{i=1}^{n} E^2\left[w_i\right]\left(E\left[y_i^2\right] + E\left[z_i^2\right]\right)^2 + \sum_{i=1}^{n}\left(E\left[I_{ox_i}^2\right] - E^2\left[I_{ox_i}\right]\right)$$

Replacing y_i by x_i, and z_i by x_i, yields the equations for I_{yy} and I_{zz} respectively.

Product of Inertia in XY Plane, I_{xy}

$$\mu_{I_{xy}} = \sum_{i=1}^{n} E\left[w_i\right] E\left[x_i\right] E\left[y_i\right] + \sum_{i=1}^{n} E\left[I_{oxy_i}\right]$$

$$\sigma_{I_{xy}}^2 = \sum_{i=1}^{n} E\left[w_i^2\right] E\left[x_i^2\right] E\left[y_i^2\right] - E^2\left[w_i\right] E^2\left[x_i\right] E^2\left[y_i^2\right]$$

$$+ \sum_{i=1}^{n}\left(E\left[I_{oxy_i}^2\right] - E^2\left[I_{oxy_i}\right]\right)$$

Replacing y_i by z_i in the above equations gives the parameters for I_{xz}.

Angle of Principal Axis Offset Angle in XY Plane, θ_{xy}

$$\mu_{\theta_{xy}} \cong \frac{\mu_{I_{xy}}}{\mu_{I_{yy}} - \mu_{I_{xx}}}$$

$$\sigma^2_{\theta_{xy}} \cong \frac{1}{\left(\mu_{I_{yy}} - \mu_{I_{xx}}\right)^2} \left(\sigma^2_{I_{xy}} + \mu^2_{\theta_{xy}} \sigma^2_{D_{xy}} - 2\mu_{\theta_{xy}} \operatorname{Cov}\left[I_{xy}, D_{xy}\right]\right)$$

where

$$\sigma^2_{D_{xy}} = \sum_{i=1}^{n} E\left[w_i^2\right] \left(E\left[x_i^4\right] - 2\,E\left[x_i^2\right] E\left[y_i^2\right] + E\left[y_i^4\right]\right)$$

$$- \sum_{i=1}^{n} E^2\left[w_i\right] \left(E\left[x_i^2\right] - E\left[y_i^2\right]\right)^2$$

$$+ \sum_{i-1}^{n} \left(E\left[I^2_{oy_i}\right] + E^2\left[I_{ox_i}\right] - E^2\left[I_{oy_i}\right] - E\left[I^2_{ox_i}\right]\right)$$

and

$$\operatorname{Cov}\left[I_{xy}, D_{xy}\right] = \sum_{i=1_{i \neq j}}^{n} \sum_{j=1}^{n} E\left[w_i\right] E\left[x_i\right] E\left[y_i\right] E\left[w_j\right] \left(E\left[x_j^2\right] - E\left[y_j^2\right]\right)$$

$$+ \sum_{i=1}^{n} E\left[w_i^2\right] \left(E\left[x_i^3\right] E\left[y_i\right] - E\left[x_i\right] E\left[y_i^3\right]\right)$$

$$+ \left(\mu_{I_{xy}} - \sum_{i=1}^{n} E\left[I_{oxy_i}\right]\right)\left(\mu_{I_{xx}} - \mu_{I_{yy}} + \sum_{i=1}^{n} E\left[I_{oy_i}\right] - \sum_{i=1}^{n} E\left[I_{ox_i}\right]\right)$$

Replacing y_i by z_i in the above equations yields the expressions for θ_{xz}.

Principal Axis Offset Angle, θ

$$\mu_\theta = \sqrt{\mu^2_{\theta_{xy}} + \mu^2_{\theta_{xz}}} + \frac{\mu^2_{\theta_{xz}} \sigma^2_{\theta_{xy}} + \mu^2_{\theta_{xy}} \sigma^2_{\theta_{xz}} - 2\mu_{\theta_{xy}} \mu_{\theta_{xz}} \operatorname{Cov}\left[\theta_{xy}, \theta_{xz}\right]}{2\left(\mu^2_{\theta_{xy}} + \mu^2_{\theta_{xz}}\right)^{3/2}}$$

$$\sigma^2_\theta = \sigma^2_{\theta_{xy}} + \sigma^2_{\theta_{xz}} + \mu^2_{\theta_{xy}} + \mu^2_{\theta_{xz}} - \mu^2_\theta$$

where

$$\operatorname{Cov}\left[\theta_{xy}, \theta_{xz}\right] \cong \frac{\mu_{H_{yz}}}{\mu_{F_{yz}}} - \mu_{\theta_{xy}} \mu_{\theta_{xz}}$$

and where

$$\mu_{H_{yz}} = \sum_{\substack{i=1 \\ i\neq j}}^{n} \sum_{j=1}^{n} E\left[w_i\right] E\left[w_j\right] E\left[x_i\right] E\left[y_i\right] E\left[x_j\right] E\left[z_j\right]$$

$$+ \sum_{i=1}^{n} E\left[w_i^2\right] E\left[x_i^2\right] E\left[y_i\right] E\left[z_i\right] + \left(\sum_{i=1}^{n} E\left[I_{oxz_i}\right]\right)\mu_{I_{xy}}$$

$$+ \left(\sum_{i=1}^{n} E\left[I_{oxy_i}\right]\right)\mu_{I_{xz}} - \left(\sum_{i=1}^{n} E\left[I_{oxz_i}\right]\right)\left(\sum_{i=1}^{n} E\left[I_{oxy_i}\right]\right)$$

and

$$
\begin{aligned}
\mu_{F_{yz}} = & \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} E\left[w_i\right]\left(E\left[x_i^2\right] - E\left[y_i^2\right]\right) E\left[w_j\right]\left(E\left[x_j^2\right] - E\left[z_j^2\right]\right) \\
& + \sum_{i=1}^{n} E\left[w_i^2\right]\left\{E\left[x_i^4\right] - E\left[x_i^2\right]\left(E\left[z_i^2\right] + E\left[y_i^2\right]\right) + E\left[y_i^2\right] E\left[z_i^2\right]\right\} \\
& + \mu_{D_{xy}}\left(\sum_{i=1}^{n} E\left[I_{oz_i}\right] - \sum_{i=1}^{n} E\left[I_{ox_i}\right]\right) + \mu_{D_{xz}}\left(\sum_{i=1}^{n} E\left[I_{oy_i}\right]\right. \\
& \left. - \sum_{i=1}^{n} E\left[I_{ox_i}\right]\right) - \left(\sum_{i=1}^{n} E\left[I_{oy_i}\right] - \sum_{i=1}^{n} E\left[I_{ox_i}\right]\right)\left(\sum_{i=1}^{n} E\left[I_{oz_i}\right] - \sum_{i=1}^{n} E\left[I_{ox_i}\right]\right)
\end{aligned}
$$

SECTION V

THE ANALYSIS

A. DISTRIBUTION OF w

By the Central Limit Theorem, for independent w_i and large enough n, we know that w is normally distributed and has a mean

$$\mu_w = E\left[\sum_{i=1}^{n} w_i\right] = \sum_{i=1}^{n} E\left[w_i\right]$$

and variance

$$\sigma_w^2 = V\left[\sum_{i=1}^{n} w_i\right] = \sum_{i=1}^{n} V\left[w_i\right]$$

or

$$\sigma_w^2 = \sum_{i=1}^{n} \left(E\left[w_i^2\right] - E^2\left[w_i\right]\right)$$

It has been shown that w converges to a normal random variable rapidly even for n as small as 3 or 4 when the $E\left[w_i\right]$ are of the same order of magnitude. A mass properties probability distribution simulation (Appendix B) of an operational reentry vehicle with $n=7$ showed the weight to be approximately normally distributed even though the $E\left[w_i\right]$ were not all of the same order of magnitude. In the simulation, the w_i were assumed uniformly distributed, which represents a conservative test for w approaching normality.

B. DISTRIBUTION OF $\overline{x}$

Let

$$t_i = w_i \; x_i$$

Since w_i and x_i are independent,

$$E\left[t_i\right] = E\left[w_i\right] \; E\left[x_i\right]$$

and

$$E\left[t_i^{\,2}\right] = E\left[w_i^{\,2}\right] \; E\left[x_i^{\,2}\right]$$

Using the identity

$$V\left[t_i\right] = E\left[t_i^{\,2}\right] - E^2\left[t_i\right]$$

we get

$$V\left[t_i\right] = E\left[w_i^{\,2}\right] E\left[x_i^{\,2}\right] - E^2\left[w_i\right] \; E^2\left[x_i\right]$$

Now let

$$t = \sum_{i=1}^{n} t_i = \sum_{i=1}^{n} w_i \; x_i$$

Since each of the t_i are independent, we have the mean of t

$$\mu_t = E\left[t\right] = E\left[\sum_{i=1}^{n} t_i\right] = \sum_{i=1}^{n} E\left[t_i\right]$$

or

$$\mu_t = \sum_{i=1}^{n} E\left[w_i\right] E\left[x_i\right]$$

and the variance of t

$$\sigma_t^2 = V\left[t\right] = V\left[\sum_{i=1}^{n} t_i\right] = \sum_{i=1}^{n} V\left[t_i\right]$$

or

$$\sigma_t^2 = \sum_{i=1}^{n} \left(E\left[w_i^2\right] E\left[x_i^2\right] - E^2\left[w_i\right] E^2\left[x_i\right]\right)$$

Expanding $\bar{x}$ as a function of t and w in a Taylor series about μ_t and μ_w respectively,

$$\bar{x} = \frac{t}{w} = \frac{\mu_t}{\mu_w} + \frac{t - \mu_t}{\mu_w} - \frac{\mu_t}{\mu_w^2} \left(w - \mu_w\right) + \ldots$$

Using only the first three terms of the expansion, the mean of $\bar{x}$ is found to be

$$\mu_{\bar{x}} = E\left[\bar{x}\right] \cong \frac{\mu_t}{\mu_w}$$

Combining the last two expressions and squaring,

$$\bar{x}^2 \cong \mu_{\bar{x}}^2 + \frac{\left(t - \mu_t\right)^2}{\mu_w^2} + \frac{\mu_{\bar{x}}^2}{\mu_w^2} \left(w - \mu_w\right)^2 - \frac{2\mu_{\bar{x}}^2}{\mu_w} \left(w - \mu_w\right)$$

$$+ \frac{2\mu_{\bar{x}}}{\mu_w} \left(t - \mu_t\right) - \frac{2\mu_{\bar{x}}}{\mu_w^2} \left(t - \mu_t\right) \left(w - \mu_w\right)$$

Taking the expected value

$$E\left[\bar{x}^2\right] \cong \mu_{\bar{x}}^2 + \frac{\sigma_t^2}{\mu_w^2} + \frac{\mu_{\bar{x}}^2}{\mu_w^2}\,\sigma_w^2 - \frac{2\mu_{\bar{x}}}{\mu_w^2}\,\mathrm{Cov}\,[t, w]$$

The variance of $\bar{x}$ is therefore

$$\sigma_{\bar{x}}^2 = E\left[\bar{x}^2\right] - E^2\,[\bar{x}]$$

$$\cong \frac{1}{\mu_w^2}\left\{\sigma_t^2 + \mu_{\bar{x}}^2\,\sigma_w^2 - 2\,\mu_{\bar{x}}\,\mathrm{Cov}\,[t, w]\right\}$$

where

$$\mathrm{Cov}\,[t, w] = E\left[\left(t - \mu_t\right)\left(w - \mu_w\right)\right]$$

$$= E\left[tw - \mu_t\,w - \mu_w\,t + \mu_t\mu_w\right]$$

$$= E\,[t\,w] - \mu_t\mu_w$$

and

$$E\,[t\,w] = E\left[\left(\sum_{i=1}^{n} w_i x_i\right)\left(\sum_{i=1}^{n} w_i\right)\right]$$

$$= E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} w_i x_i w_j\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} E\left[w_i\,x_i\,w_j\right]$$

$$= \sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n} E\left[w_i\right] E\left[x_i\right] E\left[w_j\right] + \sum_{i=1}^{n} E\left[w_i^2\right]\,E\left[x_i\right]$$

The reentry vehicle mass properties simulation (Appendix B) showed $\bar{x}$ to be approximately normal despite the dependency of the terms $\frac{x_i w_i}{w}$. The theoretical expressions above for $\mu_{\bar{x}}$ and $\sigma_{\bar{x}}^2$ agreed very well with the sample means and variances. Using y_i, z_i in the above equations, we get the equations of the means and variances of $\bar{y}$ and $\bar{z}$ respectively.

C. DISTRIBUTION OF I_{xx}

The mean of I_{xx} can be found immediately as

$$\mu_{I_{xx}} = E\left[I_{xx}\right] = E\left[\sum_{i=1}^{n} w_i\left(y_i^2 + z_i^2\right) + \sum_{i=1}^{n} I_{ox_i}\right]$$

$$= \sum_{i=1}^{n} E\left[w_i\right]\left(E\left[y_i^2\right] + E\left[z_i^2\right]\right) + \sum_{i=1}^{n} E\left[I_{ox_i}\right]$$

Assuming that the terms are all independent, we have the variance

$$\sigma_{I_{xx}}^2 = V\left[I_{xx}\right] = \sum_{i=1}^{n} V\left[w_i\left(y_i^2 + z_i^2\right)\right] + \sum_{i=1}^{n} V\left[I_{ox_i}\right]$$

where

$$V\left[w_i\left(y_i^2 + z_i^2\right)\right] = E\left[w_i^2\left(y_i^2 + z_i^2\right)^2\right] - E^2\left[w_i\left(y_i^2 + z_i^2\right)\right]$$

$$= E\left[w_i^2\right]\left(E\left[y_i^4\right] + 2E\left[y_i^2\right]E\left[z_i^2\right] + E\left[z_i^4\right]\right)$$

$$- E^2\left[w_i\right]\left(E\left[y_i^2\right] + E\left[z_i^2\right]\right)^2$$

Hence,

$$\sigma_{I_{xx}}^{\ 2} = \sum_{i=1}^{n} \Bigg\{ E\left[w_i^{\,2}\right] \left(E\left[y_i^{\,4}\right] + 2E\left[y_i^{\,2}\right] E\left[z_i^{\,2}\right] + E\left[z_i^{\,4}\right] \right)$$

$$- E^2\left[w_i\right] \left(E\left[y_i^{\,2}\right] + E\left[z_i^{\,2}\right] \right)^2 + E\left[I_{ox_i}^{\ 2}\right] - E^2\left[I_{ox_i}\right] \Bigg\}$$

Note that $\mu_{I_{xx}}$ and $\sigma_{I_{xx}}^2$ are exact. The Central Limit Theorem would guarantee that I_{xx} is asymptotically normal if the terms $w_i\left(y_i^{\,2} + z_i^{\,2}\right)$ and I_{ox_i} were independent. It was shown in the previously mentioned simulation that I_{xx} is, for all practical purposes, approximately normal. The statistical parameters for I_{xy} and I_{zz} are found similarly.

D. DISTRIBUTION OF I_{xy}

The mean of I_{xy} is

$$\mu_{I_{xy}} = E\left[I_{xy}\right] = E\left[\sum_{i=1}^{n} \left(w_i\, x_i\, y_i + I_{oxy_i} \right)\right]$$

$$= \sum_{i=1}^{n} \left(E\left[w_i\right] E\left[x_i\right] E\left[y_i\right] + E\left[I_{oxy_i}\right] \right)$$

Assuming each term independent, we find the variance of I_{xy} to be

$$\sigma_{I_{xy}}^2 = V\left[I_{xy}\right] = V\left[\sum_{i=1}^{n} \left(w_i\, x_i\, y_i + I_{oxy_i} \right)\right]$$

$$= \sum_{i=1}^{n} V\left[w_i\, x_i\, y_i + I_{oxy_i}\right]$$

or

$$\sigma_{I_{xy}}^2 = \sum_{i=1}^{n} \left(E\left[w_i^2 x_i^2 y_i^2\right] - E^2\left[w_i x_i y_i\right] + V\left[I_{oxy_i}\right]\right)$$

$$= \sum_{i=1}^{n} \left(E\left[w_i^2\right] E\left[x_i^2\right] E\left[y_i^2\right] - E^2\left[w_i\right] E^2\left[x_i\right] E^2\left[y_i\right] \right.$$

$$\left. + E\left[I_{oxy_i}^{\ 2}\right] - E^2\left[I_{oxy_i}\right]\right)$$

Again, these parameters are exact and the simulation has shown that I_{xy} is approximately normal. The parameters for I_{xz} are similar to those for I_{xy}.

E. DISTRIBUTION OF θ_{xy}

For θ_{xy} small, we can rewrite

$$\tan 2\,\theta_{xy} = \frac{2\,I_{xy}}{I_{yy} - I_{xx}}$$

as

$$\theta_{xy} \cong \frac{I_{xy}}{I_{yy} - I_{xx}}\,.$$

Now let

$$D_{xy} = I_{yy} - I_{xx} = \sum_{i=1}^{n} \left\{ w_i \left(x_i^2 - y_i^2\right) + I_{oy_i} - I_{ox_i} \right\}$$

The mean of D_{xy} is simply

$$\mu_{D_{xy}} = E\left[D_{xy}\right] = \mu_{I_{yy}} - \mu_{I_{xx}}$$

and the variance of D_{xy} is

$$\sigma_{D_{xy}}^{\;2} = V\left[D_{xy}\right] = V\left[\sum_{i=1}^{n}\left\{w_i\left(x_i^{\;2} - y_i^{\;2}\right) + I_{oy_i} - I_{ox_i}\right\}\right]$$

Assuming the three terms are independent,

$$\sigma_{D_{xy}}^{\;2} = \sum_{i=1}^{n}\left(V\left[w_i\left(x_i^{\;2} - y_i^{\;2}\right)\right] + V\left[I_{oy_i}\right] - V\left[I_{ox_i}\right]\right)$$

or

$$\begin{aligned}\sigma_{D_{xy}}^{\;2} = \sum_{i=1}^{n}\Bigg\{ & E\left[w_i^{\;2}\right]\left(E\left[x_i^{\;4}\right] - 2\,E\left[x_i^{\;2}\right]E\left[y_i^{\;2}\right] + E\left[y_i^{\;4}\right]\right) \\ & - E^2\left[w_i\right]\left(E\left[x_i^{\;2}\right] - E\left[y_i^{\;2}\right]\right)^2 \\ & + E\left[I_{oy_i}^{\;2}\right] - E^2\left[I_{oy_i}\right] - E\left[I_{ox_i}^{\;2}\right] + E^2\left[I_{ox_i}\right]\Bigg\}\end{aligned}$$

Expanding θ_{xy} as a function of I_{xy} and D_{xy} in a Taylor series about the points I_{xy} and D_{xy} respectively,

$$\theta_{xy} = \frac{I_{xy}}{D_{xy}} = \frac{\mu_{I_{xy}}}{\mu_{D_{xy}}} + \frac{\left(I_{xy} - \mu_{I_{xy}}\right)}{\mu_{D_{xy}}} - \frac{\mu_{I_{xy}}}{\mu_{D_{xy}}^{\;2}}\left(D_{xy} - \mu_{D_{xy}}\right) + \ldots$$

The mean of θ_{xy} is

$$\mu_{\theta_{xy}} \cong \frac{\mu_{I_{xy}}}{\mu_{D_{xy}}}$$

The variance of θ_{xy} is found by a method similar to that for $\sigma_{\overline{x}}^2$,

$$\sigma_{\theta_{xy}}^2 = \frac{1}{\mu_{D_{xy}}^2} \left\{ \sigma_{I_{xy}}^2 + \mu_{\theta_{xy}}^2 \sigma_{D_{xy}}^2 - 2\mu_{\theta_{xy}} \mathrm{Cov}\left[I_{xy}, D_{xy}\right] \right\}$$

where

$$\mathrm{Cov}\left[I_{xy}, D_{xy}\right] = E\left[I_{xy} D_{xy}\right] - \mu_{I_{xy}} \mu_{D_{xy}}$$

and where

$$\begin{aligned}
E\left[I_{xy} D_{xy}\right] \cong\ & E\left[\left(\sum_{i=1}^{n} w_i x_i y_i + \sum_{i=1}^{n} I_{oxy_i}\right)\left(\sum_{i=1}^{n} w_i \left(x_i^2 - y_i^2\right)\right.\right. \\
& \left.\left. + \sum_{i=1}^{n} I_{oy_i} - \sum_{i=1}^{n} I_{ox_i}\right)\right] \\
=\ & E\left[\left\{\sum_{i=1}^{n} w_i x_i y_i\right\}\left\{\sum_{i=1}^{n} w_i \left(x_i^2 - y_i^2\right)\right\}\right. \\
& + \left\{\sum_{i=1}^{n} w_i x_i y_i\right\}\left\{\sum_{i=1}^{n} I_{oy_i} - \sum_{i=1}^{n} I_{ox_i}\right\} \\
& + \left\{\sum_{i=1}^{n} w_i \left(x_i^2 - y_i^2\right)\right\}\left\{\sum_{i=1}^{n} I_{oxy_i}\right\} \\
& \left. + \left\{\sum_{i=1}^{n} I_{oxy_i}\right\}\left\{\sum_{i=1}^{n} I_{oy_i} - \sum_{i=1}^{n} I_{ox_i}\right\}\right]
\end{aligned}$$

Assuming the component moments and products of inertia are independent, we have

$$\mathrm{Cov}\left[I_{xy}, D_{xy}\right] \cong \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} E\left[w_i\right] E\left[x_i\right] E\left[y_i\right] E\left[w_j\right] \left(E\left[x_j^2\right] - E\left[y_j^2\right]\right)$$

$$+ \sum_{i=1}^{n} E\left[w_i^2\right] \left(E\left[x_i^3\right] E\left[y_i\right] - E\left[x_i\right] E\left[y_i^3\right]\right)$$

$$+ \left(\mu_{I_{xy}} - \sum_{i=1}^{n} E\left[I_{oxy_i}\right]\right)\left(\mu_{I_{xx}} - \mu_{I_{yy}} + \sum_{i=1}^{n} E\left[I_{oy_i}\right] - \sum_{i=1}^{n} E\left[I_{ox_i}\right]\right)$$

Replacing y by z in the above equations yields the parameters for θ_{xz}.

F. DISTRIBUTION OF θ

For small values of θ, we can rewrite

$$\theta = \tan^{-1}\left(\sqrt{\tan^2 \theta_{xy} + \tan^2 \theta_{xz}}\right)$$

as

$$\theta \cong \sqrt{\theta_{xy}^2 + \theta_{xz}^2}$$

Then, θ can be expanded in a Taylor series about the points $\mu_{\theta_{xy}}$ and $\mu_{\theta_{xz}}$:

$$\theta \cong \sqrt{\mu_{\theta_{xy}}^2 + \mu_{\theta_{xz}}^2} + \frac{\left(\theta_{xy} - \mu_{\theta_{xz}}\right) \mu_{\theta_{xy}}}{\sqrt{\mu_{\theta_{xy}}^2 + \mu_{\theta_{xz}}^2}} +$$

$$+ \frac{\left(\theta_{xz} - \mu_{\theta_{xz}}\right) \mu_{\theta_{xz}}}{\sqrt{\mu^2_{\theta_{xy}} + \mu^2_{\theta_{xz}}}} + \frac{1}{2} \frac{\mu^2_{\theta_{xz}} \left(\theta_{xy} - \mu_{\theta_{xy}}\right)^2}{\left(\mu^2_{\theta_{xy}} + \mu^2_{\theta_{xz}}\right)^{3/2}}$$

$$- \frac{\mu_{\theta_{xy}} \mu_{\theta_{xz}} \left(\theta_{xy} - \mu_{\theta_{xz}}\right) \left(\theta_{xz} - \mu_{\theta_{xz}}\right)}{\left(\mu^2_{\theta_{xy}} + \mu^2_{\theta_{xz}}\right)^{3/2}}$$

$$+ \frac{1}{2} \frac{\mu_{\theta_{xy}}{}^2 \left(\theta_{xz} - \mu_{\theta_{xz}}\right)^2}{\left(\mu_{\theta_{xy}}{}^2 + \mu_{\theta_{xz}}{}^2\right)^{3/2}} + \ldots$$

Taking the expected value of the first and second-order terms,

$$\mu_\theta = E\left[\theta\right] \cong \sqrt{\mu_{\theta_{xy}}{}^2 + \mu_{\theta_{xz}}{}^2} + \frac{\mu_{\theta_{xz}}{}^2 \sigma_{\theta_{xy}}{}^2 + \mu_{\theta_{xy}}{}^2 \sigma_{\theta_{xz}}{}^2}{2\left(\mu_{\theta_{xy}}{}^2 + \mu_{\theta_{xz}}{}^2\right)^{3/2}}$$

$$- \frac{\mu_{\theta_{xy}} \mu_{\theta_{xz}} \operatorname{Cov}\left[\theta_{xy}, \theta_{xz}\right]}{\left(\mu_{\theta_{xy}}{}^2 + \mu_{\theta_{xz}}{}^2\right)^{3/2}}$$

or

$$\mu_\theta \cong \sqrt{\mu^2_{\theta_{xy}} + \mu^2_{\theta_{xz}}} + \frac{\mu^2_{\theta_{xz}} \sigma^2_{\theta_{xy}} + \mu^2_{\theta_{xy}} \sigma^2_{\theta_{xz}} - 2\mu_{\theta_{xy}} \mu_{\theta_{xz}} \operatorname{Cov}\left[\theta_{xy}, \theta_{xz}\right]}{\left(\mu^2_{\theta_{xy}} + \mu^2_{\theta_{xz}}\right)^{3/2}}$$

Let

$$I_{xy} I_{xz} = H_{yz}$$

and

$$D_{xy} D_{xz} = F_{xz}$$

We can then write

$$\theta_{xy} \theta_{xz} = \left(\frac{I_{xy}}{D_{xy}}\right)\left(\frac{I_{xz}}{D_{xz}}\right) = \frac{H_{yz}}{F_{yz}}$$

The first-order approximation for the mean of $\theta_{xy} \; \theta_{xz}$ is

$$E\left[\theta_{xy} \; \theta_{xz}\right] = \frac{\mu_{H_{yz}}}{\mu_{F_{yz}}}$$

where

$$\mu_{H_{yz}} = \sum_{\substack{i \neq 1 \\ i \neq j}}^{n} \sum_{j=1}^{n} E\left[w_i\right] E\left[x_i\right] E\left[y_i\right] E\left[w_j\right] E\left[x_j\right] E\left[z_j\right]$$

$$+ \sum_{i=1}^{n} E\left[w_i^2\right] E\left[x_i^2\right] E\left[y_i\right] E\left[z_i\right]$$

$$+ \left(\sum_{i=1}^{n} E\left[I_{oxz_i}\right]\right)\mu_{I_{xy}} + \left(\sum_{i=1}^{n} E\left[I_{oxy_i}\right]\right)\mu_{I_{xz}}$$

$$- \left(\sum_{i=1}^{n} E\left[I_{oxz_i}\right]\right)\left(\sum_{i=1}^{n} E\left[I_{oxy_i}\right]\right)$$

and

$$\mu_{F_{yz}} = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} E\left[w_i\right] \left(E\left[x_i^2\right] - E\left[y_i^2\right]\right) E\left[w_j\right]\left(E\left[x_j^2\right] - E\left[z_j^2\right]\right)$$

$$+ \sum_{i=1}^{n} E\left[w_i^2\right]\left\{E\left[x_i^4\right] - E\left[x_i^2\right]\left(E\left[z_i^2\right] + E\left[y_i^2\right]\right) + E\left[y_i^2\right] E\left[z_i^2\right]\right\}$$

$$+ \mu_{D_{xy}} \sum_{i=1}^{n} \left(E\left[I_{oz_i}\right] - \sum_{i=1}^{n} E\left[I_{ox_i}\right]\right)$$

$$+ \mu_{D_{xz}} \sum_{i=1}^{n} \left(E\left[I_{oy_i}\right] - \sum_{i=1}^{n} E\left[I_{ox_i}\right]\right)$$

$$- \left(\sum_{i=1}^{n} E\left[I_{oy_i}\right] - \sum_{i=1}^{n} E\left[I_{ox_i}\right]\right)\left(\sum_{i=1}^{n} E\left[I_{oz_i}\right] - \sum_{i=1}^{n} E\left[I_{ox_i}\right]\right)$$

The covariance term can then be computed as

$$\mathrm{Cov}\left[\theta_{xy}, \theta_{xz}\right] = E\left[\theta_{xy}\, \theta_{xz}\right] - \mu_{\theta_{xy}}\, \mu_{\theta_{xz}}$$

In some cases the computation of μ_θ may be considerably simplified by noting that

$$\mathrm{Cov}\left[\theta_{xy}, \theta_{xz}\right] = \rho\, \sigma_{\theta_{xy}}\, \sigma_{\theta_{xz}}$$

where ρ is the correlation coefficient between θ_{xy} and θ_{xz}, and is bounded,

$$|\rho| \leq 1$$

If it happens that

$$\left| \mu^2_{\theta_{xz}} \sigma^2_{\theta_{xz}} + \mu^2_{\theta_{xy}} \sigma^2_{\theta_{xz}} \right| >> 2 \left| \mu_{\theta_{xy}} \mu_{\theta_{xz}} \sigma_{\theta_{xy}} \sigma_{\theta_{xz}} \right|$$

(i.e., twice the magnitude of the second term is negligible compared to the magnitude of the first term), then we can write

$$\left| \mu^2_{\theta_{xz}} \sigma^2_{\theta_{xy}} + \mu^2_{\theta_{xy}} \sigma^2_{\theta_{xz}} \right| >> \left| 2\rho\mu_{\theta_{xy}} \mu_{\theta_{xz}} \sigma_{\theta_{xy}} \sigma_{\theta_{xz}} \right|$$

or

$$\left| \mu^2_{\theta_{xz}} \sigma^2_{\theta_{xy}} + \mu^2_{\theta_{xy}} \sigma^2_{\theta_{xz}} \right| >> \left| 2\mu_{\theta_{xy}} \mu_{\theta_{xz}} \operatorname{Cov}\left[\theta_{xy}, \theta_{xz}\right] \right|$$

and μ_θ is written neglecting the covariance term

$$\mu_\theta \cong \sqrt{\mu^2_{\theta_{xy}} + \mu^2_{\theta_{xz}}} + \frac{\mu^2_{\theta_{xz}} \sigma^2_{\theta_{xy}} + \mu^2_{\theta_{xy}} \sigma^2_{\theta_{xz}}}{\left(\mu^2_{\theta_{xy}} + \mu^2_{\theta_{xz}}\right)^{3/2}}$$

The variance for θ may be readily found since we have

$$E\left[\theta_{xy}{}^2\right] = \sigma_{\theta_{xy}}{}^2 + \mu_{\theta_{xy}}{}^2$$

and

$$E\left[\theta_{xz}{}^2\right] = \sigma_{\theta_{xz}}{}^2 + \mu_{\theta_{xz}}{}^2$$

The variance is

$$\sigma_\theta^2 = E\left[\theta^2\right] - E^2\left[\theta\right] = E\left[\theta_{xy}^2 + \theta_{xz}^2\right] - \mu_\theta^2$$

or

$$\sigma_\theta^2 = \sigma_{\theta_{xy}}^2 + \sigma_{\theta_{xz}}^2 + \mu_{\theta_{xy}}^2 + \mu_{\theta_{xz}}^2 - \mu_\theta^2$$

The theoretical parameters for θ agreed quite well with those found by simulation for the reentry vehicle. The distribution function was shown to be very nearly normal.

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SECTION VI

CONCLUSION

Any sum of independent, normal, random variables is normally distributed. For large n , the Central Limit Theorem states that the sum is approximately normal regardless of the forms of the independent component distributions. The largeness of n must be measured primarily as a function of those components which dominate. Deviation from the theoretical normal will also occur if the terms are dependent. If the normal model is inadequate for a particular problem, a simulation may be performed to establish the general form of the distribution function. In the absence of other information, the simulation shows that the normal distribution is a reasonably good first approximation.

The derived means and variances were verified (Appendix B). Although some of the equations seem formidable, they involve only simple algebraic manipulations and therefore are easily programmed for computer analysis. The equations yield good approximations of the true means and variances and are valid for all possible component and system distributions.

The form of the probability density function used in conjunction with the derived mean and variance completely describes the statistical nature of the mass property in question. Care must be used in making probability statements, however, because the density function, the derived parameters, and especially the input distributions are really estimates of reality. For example, the simulation used components which were uniformly distributed. It would be impossible to make a 3σ probability statement about the weight based on the theoretical normal because $\mu_W + 3\sigma_W$ exceeds the largest possible value w may have. The reason for this is that the input components are uniform and therefore have finite upper bounds. If we had assumed, instead, that the components were normal, then the 3σ statement would have been exact.

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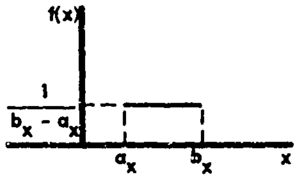
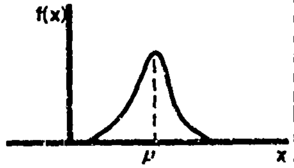
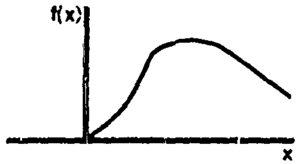
APPENDIX A

DISTRIBUTIONS AND MOMENTS OF COMMONLY USED PROBABILITY LAWS

Probability Law	Density Function	$E[x]$ (mean)	$E[x^2]$ (second moment)	$E[x^3]$ (third moment)	$E[x^4]$ (fourth moment)	$E[(x - E[x])^2]$ (variance)
Uniform 	$f(x) = \frac{1}{b-a}$, $a \le x \le b$ $= 0$, otherwise	$\frac{a+b}{2}$	$\frac{b^2 + ab + a^2}{3}$	$\frac{b^4 - a^4}{4(b-a)}$	$\frac{b^5 - a^5}{5(b-a)}$	$\frac{(b-a)^2}{12}$
Normal 	$f(x) = \frac{1}{\sqrt{2\Pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, $-\infty < x < \infty$	μ	$\mu^2 + \sigma^2$	$\mu^3 + \sigma^2\mu$	$\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$	σ^2
Gamma 	$f(x) = \frac{\lambda}{\Gamma(r)} (\lambda x)^{r-1} e^{-\lambda x}$, $x > 0$ $= 0$, otherwise	$\frac{r}{\lambda}$	$\frac{r(r+1)}{\lambda^2}$	$\frac{r(r+1)(r+2)}{\lambda^3}$	$\frac{r(r+1)(r+2)(r+3)}{\lambda^4}$	$\frac{r}{\lambda^2}$

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APPENDIX B

MASS PROPERTIES SIMULATION

In order to verify the theoretically derived expressions for mean and variance and to gain some insight into the general forms of the probability density functions, a simulation was performed. The object considered is a typical reentry vehicle subdivided into seven components. Because the simulation was performed by hand, the component mass properties were assumed to be uniformly distributed. For each component mass property, a sample of size 100 was generated according to its prescribed probability law. The system equations were then used to generate samples of size 100 for each system mass property. Following this, estimates for the means and variances were derived and compared to the theoretical results of the same model. Minimum variance unbiased estimates were used. For the mean and variance, they are of the form

$$\hat{u} = \frac{1}{m} \sum_{j=1}^{m} x_j$$

and

$$\hat{\sigma}^2 = \frac{1}{m-1} \sum_{j=1}^{m} \left(x_j - \hat{u}\right)^2$$

respectively. Here x_j represents the j^{th} sample of the arbitrary mass property, x. The sample size is m. In order to display the verification of the theoretical parameters without releasing classified information, the following criterion was used.

The estimates $\hat{u}$ and $\hat{\sigma}^2$ are random variables with approximate standard deviations as follows:

$$SD[\hat{u}] \cong \frac{\hat{\sigma}}{\sqrt{m}}$$

$$SD\left[\hat{\sigma}^2\right] \cong \sqrt{\frac{2}{m-1}}\ \hat{\sigma}^2$$

The approximation results from the fact that the point estimate of $\hat{\sigma}^2$ is used in place of the unknown variance. The expressions

$$\frac{\left|\hat{\sigma}^2 - \sigma^2\right|}{SD[\hat{\sigma}^2]}$$

and

$$\frac{\left|\hat{u} - u\right|}{SD[\hat{u}]}$$

give a measure of the difference between theory and simulation in terms of the standard deviations (inherent errors) of the estimates. These results for the simulation are listed in Figure B-1.

Distribution functions resulting from the simulation are plotted in Figures B-2 through B-7. They are transformed to zero mean and unit variance for security reasons. Theoretical normal curves with zero mean and unit variance are superimposed for comparison.

	$\frac{\lvert \hat{\sigma}^2 - \sigma^2 \rvert}{SD[\hat{\sigma}^2]}$	$\frac{\lvert \hat{\mu} - \mu \rvert}{SD[\hat{\mu}]}$
w	1.26	0.61
$\bar{x}$	1.27	0.59
I_{xx}	0.33	2.27
I_{xy}	0.36	0.04
θxy	0.24	0.06
θ	0.10	0.07

Figure B-1. Difference between Theoretical and Simulated Parameters in Terms of Standard Deviations of the Estimators

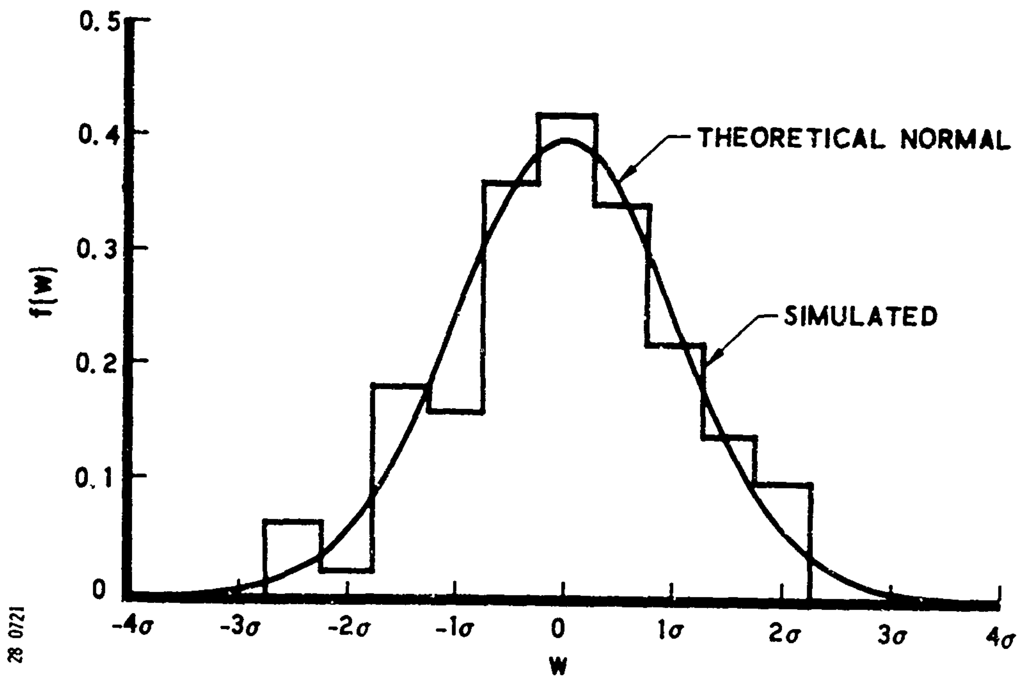

Figure B-2. Nondimensionalized Density Function of Weight (w)

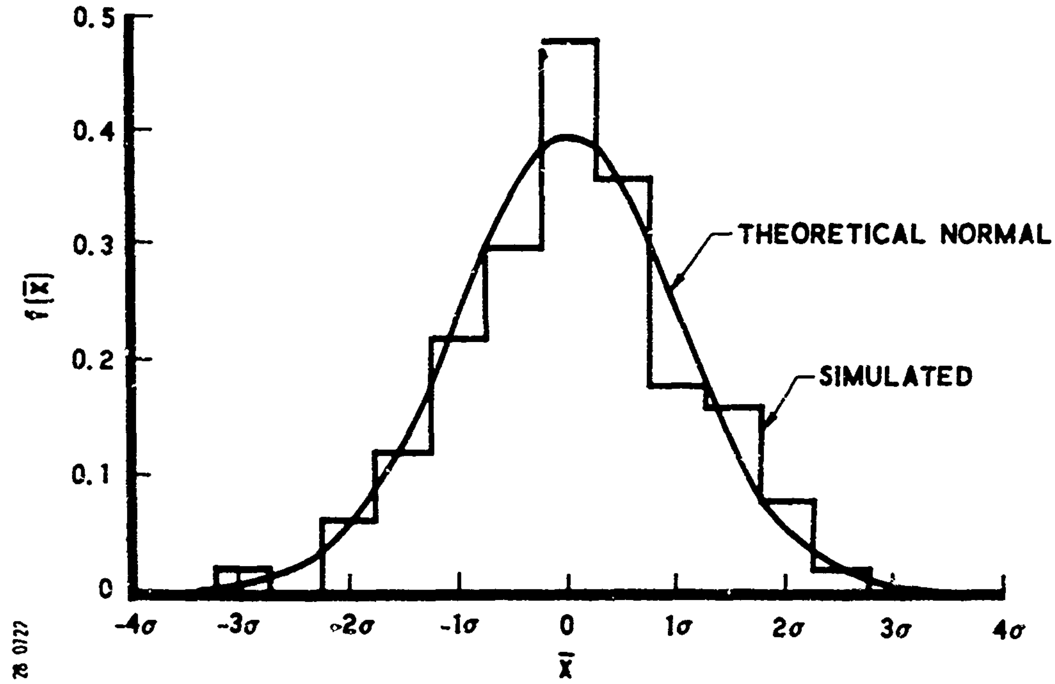

Figure B-3. Nondimensionalized Density Function of Longitudinal Center of Gravity ($\bar{x}$)

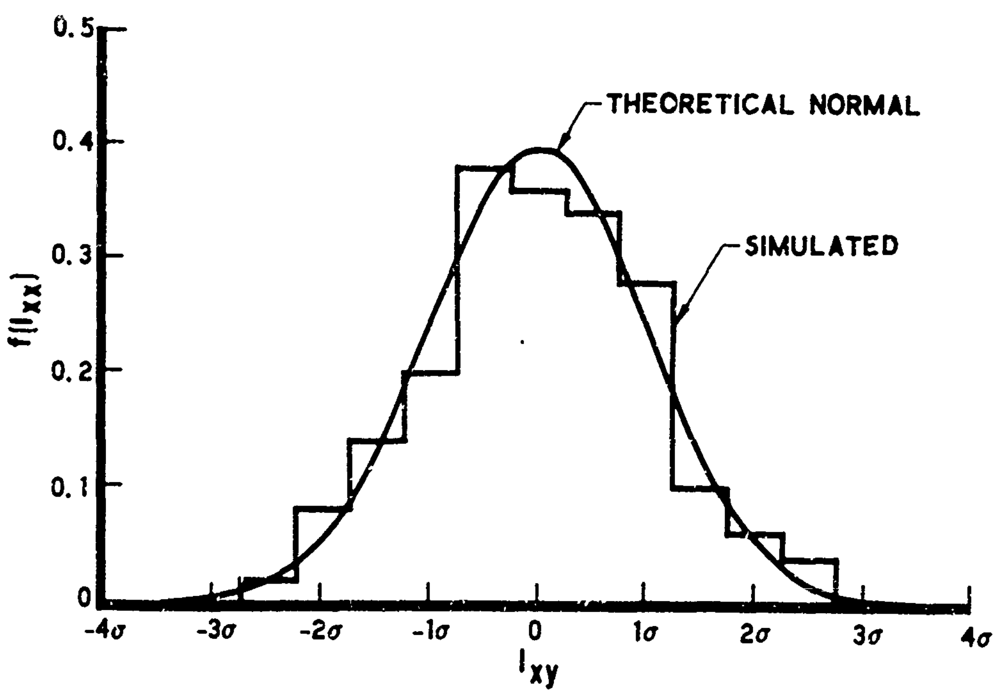

Figure B-4. Nondimensionalized Density Function of Moment of Inertia (I_{xx})

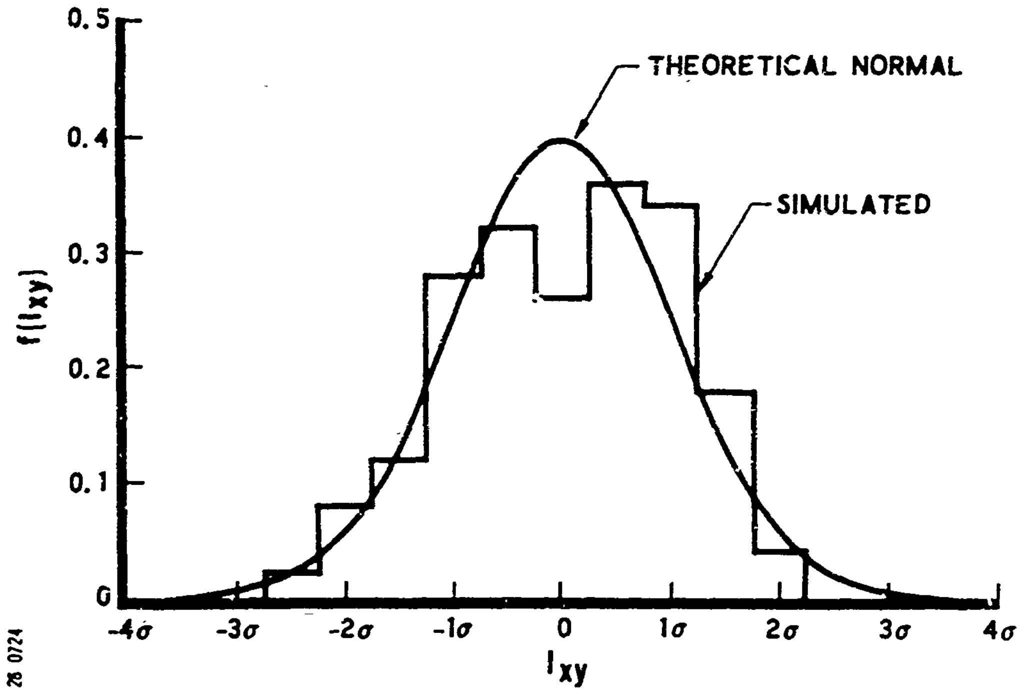

Figure B-5. Nondimensionalized Density Function of Product of Inertia (I_{xy})

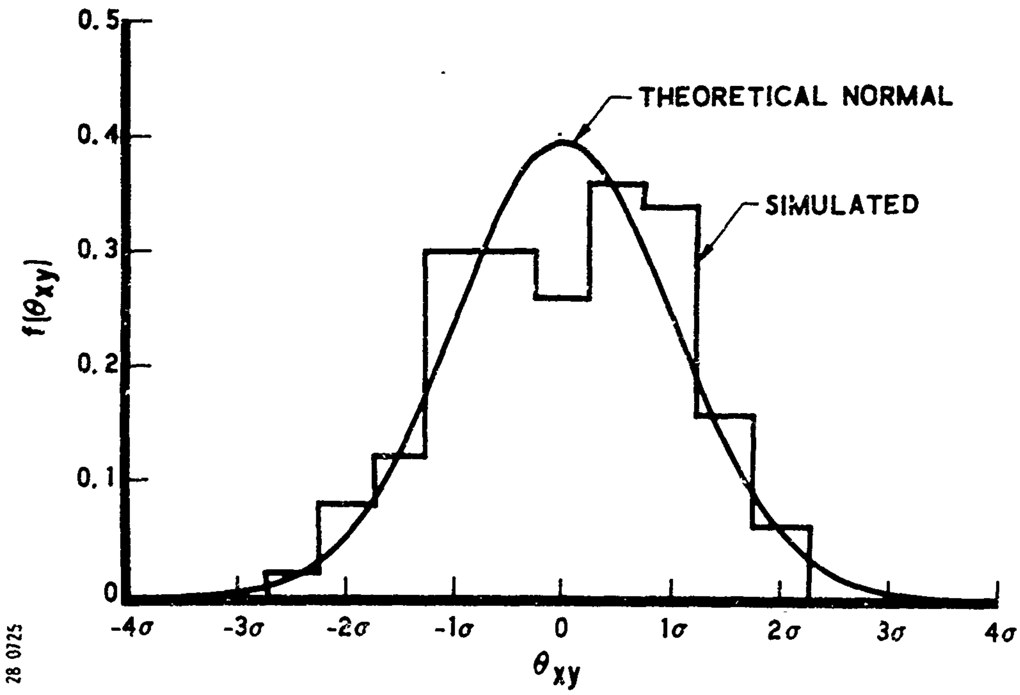

Figure B-6. Nondimensionalized Density Function of Principal Axis Offset Projection (θ_{xy})

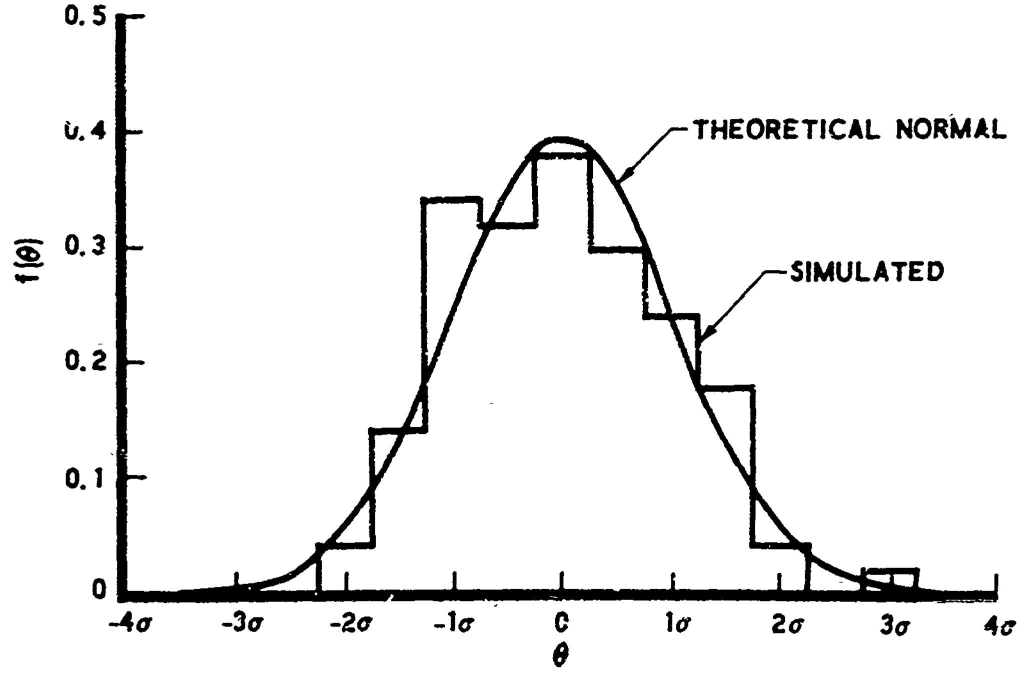

Figure B-7. Nondimensionalized Density Function of Principal Axis Offset Angle (θ)

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DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) The Aerospace Corporation San Bernardino, California	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b GROUP --

3. REPORT TITLE

A METHOD FOR DETERMINING PROBABILITY DISTRIBUTIONS FOR MASS PROPERTIES OF SYSTEMS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report

5. AUTHOR(S) (Last name, first name, initial)

Bond, Frank C.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
June 1968	50	5

8a. CONTRACT OR GRANT NO. F04695-67-C-0158	9a. ORIGINATOR'S REPORT NUMBER(S) TR-0158(S3960-30)-2
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	SAMSO-TR-68-251

10. AVAILABILITY/LIMITATION NOTICES

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11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
--	Space and Missile Systems Organization Air Force Systems Command Norton Air Force Base, California 92409

13. ABSTRACT

An analysis is performed to develop expressions for the statistical parameters of system mass properties. It is assumed that the component weights, centers of gravity, and inertias have known probability distributions. The general forms of the density functions generated are deduced from the calculus of probability whenever applicable. Means and variances of the system mass properties are computed from linear functions of the statistical parameters of the components. The input random variables may be distributed according to any probability law or combination of probability laws for which the moments are known. From conceptual design to detailed specification drawings, the probability laws governing mass properties estimation change. This method may be utilized at any point in the life cycle of system design as it is independent of the particular forms of the distributions used as input.

14. KEY WORDS

Mass properties statistical analysis
Statistical analysis
Mass properties probability distributions
Mass properties means and variances

Abstract (Continued)

AEROSPACE CORPORATION

ERRATA NOTICE

TO: Copyholder   CC. SBPUB   DATE: 15 August 1968

SUBJECT: Aerospace Report No. TR-0158(S3960-30)-2
Air Force Report No. SAMSO-TR-68-251
(SMQVB, Norton Air Force Base)
TITLE: A Method for Determining Probability Distributions for Mass Properties of Systems (Unclassified Report)

FROM: Publications

AD672760

The following errata occurring in subject document should be corrected by the copyholder.

Page 12, line 8 (third equation, last term):

$$+ \sum_{\overset{i=1}{\cancel{i-1}}}^{n} \left(E\left[I_{oy_i}^2\right] \overset{-}{\cancel{+}} E^2\left[I_{ox_i}\right] - E^2\left[I_{oy_i}\right] \overset{+}{\cancel{-}} E\left[I_{ox_i}^2\right] \right)$$

Page 22, lines 4 and 8 (second equation and third equation, last term):

$$\sigma_{D_{xy}}^{\ 2} = \sum_{i=1}^{n} \left(V\left[w_i \left(x_i^{\ 2} - y_i^{\ 2}\right)\right] + V\left[I_{oy_i}\right] \overset{+}{\cancel{-}} V\left[I_{ox_i}\right] \right)$$

$$+ E\left[I_{oy_i}^{\ 2}\right] - E^2\left[I_{oy_i}\right] \overset{+}{\cancel{-}} E\left[I_{ox_i}^{\ 2}\right] \overset{-}{\cancel{+}} E^2\left[I_{ox_i}\right] \Big\}$$

Page 27, line 13 (last line):

$$|\rho| \leq 1.0$$

Page 31, lines 22, 23, and 26:

3σ should read 4σ in each case.